JOURNAL OF ECONOMIC THEORY 45, 288–315 (1988)

Intertemporal Optimality in a Closed Linear Model of Production*

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Received June 23, 1986; revised March 18, 1988

The paper presents the main results on intertemporal optimality with discounting in a closed linear model of production, including the price characterization of optimal programs, the existence of a steady-state optimal program, and a turnpike property of optimal programs from arbitrary initial stocks. Some of these results are used to provide a characterization of the optimality of competitive programs in terms of a "decentralizable" condition. *Journal of Economic Literature* Classification Number: 111.

1. INTRODUCTION

The purpose of this paper is twofold. First, it tries to present the main results on intertemporal optimality of infinite horizon programs in a simple closed linear model of production. Second, it presents results on the characterization of the optimality of competitive programs in terms of a "decentralizable" condition (Theorems 1 and 2, Section 5).

Regarding the first aspect, we note that the results that we present are more or less well-known in the literature, particularly in the papers by McFadden [8] and Atsumi [1]. Our purpose is to present a systematic

* Research of the second author was supported by a National Science Foundation grant. Research on this paper was started while the first author was on sabbatic leave at Cornell University and Instituto Torcuato Di Tella, Buenos Aires. We are indebted to Mukul Majumdar for introducing us to the problems of intertemporal decentralization which this paper addresses. The present version has benefited from the detailed comments of two referees and an associate editor of the journal.

0022-0531/88 $3.00

treatment in the context of a simple closed linear model, so that the main results are highlighted, while the additional complications which arise in more general models are avoided. These considerations dictate the choice of our production framework given by the simple linear model presented in Gale [5] and the preference framework given by iso-elastic utility functions as in Atsumi [1].

After discussing a von Neumann equilibrium in Section 3, we present the main results on optimal programs in Section 4. We first establish the existence of optimal programs using the existence criterion of Atsumi [1] and Brock and Gale [2]. We then present a "price characterization" of optimal programs: a feasible program is shown to be optimal if and only if it is competitive and it satisfies the transversality condition that the value of input stocks converges to zero. The necessity side of this result is contained in McFadden [8]. However, given our simple framework, we give an alternative proof, which exploits heavily the characteriation of *efficient* programs provided by Majumdar [9] and uses only the finite-dimensional separation theorem. Next, we obtain a result on the existence of a steady-state optimal program, that is, a program along which all outputs grow at a constant growth factor, and which is optimal in the usual sense. The line of argument used is essentially constructive, following Atsumi [1], but differentiability assumptions are not used on the wèlfare function. Finally, we use all of the above results to provide a "turnpike theorem" for optimal programs. That is, output produced along an optimal program is shown to be (a) of the same "composition" asymptotically as the steady-state optimal program and (b) growing at the same growth factor asymptotically as the steady-state optimal program.

Regarding the second aspect, we note that the problem here is to characterize the optimality of competitive programs in terms of a condition which can be verified under a decentralized system. A detailed discussion of and motivation for this problem can be found in Brock and Majumdar [3]. It suffices for our purpose to note that we want to replace the transversality condition in the price characterization results of Section 4 by a period-by-period condition, which involves for any given period the prices and quantities of the given competitive program and of the steady-state program for that period. We find that the condition used by Brock and Majumdar [3]—namely, that the scalar product of the price difference and the quantity difference be non-positive at each date—works for the closed linear model as well, but with one important qualification. The steady-state optimal stock is determined only up to positive scalar multiplication. If we fix the steady-state optimal stock (by suitable normalization), we must agree to compare it with competitive programs starting only from a certain set of initial stocks (the set being dependent on the choice of the steady-state optimal stock). The line of proof of the characterization result suggests that without this qualification, it is to be expected that the result (Theorem 1) would fail. We confirm this by studying a concrete example in the one-good version of our linear model (see Example 1, Section 5).

Proofs of our results are discussed only in Section 6.

2. Preliminaries

2a. *Notation*

Let R^n be an n-dimensional real space. For x, y in R^n, $x \geqslant y$ means $x_i \geqslant y_i$ for $i = 1, \ldots, n$; $x > y$ means $x \geqslant y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, \ldots, n$. We denote the set $\{x \text{ in } R^n : x \geqslant 0\}$ by R^n_+, and the set $\{x \text{ in } R^n : x \gg 0\}$ by R^n_{++}.

We use the sum-norm on R^n; that is, the norm on R^n (denoted by $\|\cdot\|$) is defined by

$$\|x\| = \sum_{i=1}^{n} |x_i| \qquad \text{for all } x \text{ in } R^n.$$

The vector $(1, 1, \ldots, 1)$ in R^n is denoted by e. The ith unit vector in R^n is denoted by e^i, i.e., $e^i_j = 0$ for $i \neq j$ and $e^i_i = 1$; $i = 1, \ldots, n$, $j = 1, \ldots, n$. For any z in R^n, we denote $\min_i z_i$ by $m(z)$ and $\max_i z_i$ by $M(z)$.

Let A be an $n \times n$ real matrix. The generic element of A is denoted by a_{ij}, $i = 1, \ldots, n$ and $j = 1, \ldots, n$. The matrix, A, is called non-negative, and written as $A \geqslant 0$, if $a_{ij} \geqslant 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. It is called positive, and written as $A > 0$, if $A \geqslant 0$ and A is not the null matrix. It is called strictly positive, and written as $A \gg 0$, if $a_{ij} > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

2b. *The Model*

Consider an economy described by (Ω, w, δ), where Ω, a subset of $R^n_+ \times R^n_+$, is the technology set, $w: R^n_+ \to R$ is a welfare function, and δ is a discount factor, satisfying $0 < \delta < 1$.

We consider the technology to be of a simple polyhedral type, as specified in Gale [5],

$$\Omega = \{(x, y) \text{ in } R^n_+ \times R^n_+ : Ay \leqslant x\},$$

where A is an $n \times n$ real matrix, satisfying

(A.1) A is strictly positive; that is, $a_{ij} > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

(A.2) A is productive; that is, there is y^0 in R^n_+ such that

$$Ay^0 \ll y^0.$$

Assumption (A.1) can be weakened somewhat using the theory of "primitive" matrices as discussed in Nikaido [10, pp. 108–114]. Assumption (A.2) is equivalent to the Hawkins–Simon condition (Nikaido [10, p. 90]).

Following Atsumi [1], we assume that the welfare function is of an iso-elastic type:

(A.3) $w(c)=[f(c)]^{1-\alpha}$ for c in R^n_+ where $0<\alpha<1$, and $f\colon R^n_+ \to R_+$ satisfies the following restrictions:

(a) f is concave and continuous on R^n_+.

(b) f is homogeneous of degree one.

(c) $f(c')\geqslant f(c)$ when $c'\geqslant c$; $f(c')>f(c)$ if $c'>c$ and $f(c)>0$.

(d) $f(c)>0$ for $c\gg 0$.

Remark 1. $w(c)\geqslant 0$ for c in R^n_+, since $f(c)\geqslant 0$ for c in R^n_+. Furthermore, from (A.3) (a) and (b) it follows that $f(0)=0$. Hence, $w(0)=0$.

A *program* from $\tilde{y}$ in R^n_+ is a sequence $\langle x(t), y(t)\rangle$, such that $y(0)=\tilde{y}$, $0\leqslant x(t)\leqslant y(t)$, and $(x(t), y(t+1))$ is in Ω for $t\geqslant 0$. Associated with a program $\langle x(t), y(t)\rangle$ is a *consumption sequence* $\langle c(t)\rangle$ given by

$$c(t)=y(t)-x(t) \qquad \text{for} \quad t\geqslant 0.$$

A program $\langle x(t), y(t)\rangle$ from $\tilde{y}$ is a *steady-state program* if $\tilde{y}>0$ and there is a real number, $g>0$, such that

$$y(t+1)=gy(t) \qquad \text{for} \quad t\geqslant 0.$$

A program $\langle x^*(t), y^*(t)\rangle$ from $\tilde{y}$ is an *optimal program* if

$$\sum_{t=0}^{\infty} \delta^t w(c^*(t))\geqslant \sum_{t=0}^{\infty} \delta^t w(c(t))$$

for all programs $\langle x(t), y(t)\rangle$ from $\tilde{y}$. A program $\langle x(t), y(t)\rangle$ from $\tilde{y}$ is a *steady-state optimal program* if it is a steady-state program and it is an optimal program from $\tilde{y}$. If there is a steady–state optimal program from $\tilde{y}$, we call $\tilde{y}$ a *steady-state optimal stock*.

A *competitive program* is a sequence $\langle x(t), y(t), p(t)\rangle$, such that

(1) $\langle x(t), y(t)\rangle$ is a program;

(2) $p(t)$ is in R^n_+ for $t\geqslant 0$;

(3) $\delta^t w(c(t))-p(t)\,c(t)\geqslant \delta^t w(c)-p(t)\,c \qquad$ for all c in R^n_+, $t\geqslant 0 \quad$ (2.1)

$p(t+1)\,y(t+1)-p(t)\,x(t)\geqslant p(t+1)\,y-p(t)\,x$
for all (x, y) in Ω, $t\geqslant 0$. $\quad$ (2.2)

A competitive program $\langle x(t), y(t), p(t)\rangle$ is said to satisfy the *transversality condition* if

$$\lim_{t\to\infty} p(t)\,x(t)=0. \tag{2.3}$$

A program $\langle \bar{x}(t), \bar{y}(t)\rangle$ from $\tilde{y}$ is called *inefficient* if there is a program $\langle x(t), y(t)\rangle$ from $\tilde{y}$, such that $c(t)\geqslant \bar{c}(t)$ for all $t\geqslant 0$, and $c(t)>\bar{c}(t)$ for some $t\geqslant 0$. It is called *efficient* if it is not inefficient.

An obvious property of an efficient program $\langle x(t), y(t)\rangle$ is that for $t\geqslant 0$, $x(t)=Ay(t+1)$.

A useful characterization of the set of (feasible) programs and the set of efficient programs is provided in Theorem 4.1 of Majumdar [9]. We state it here for ready reference.

Result 1. *If* $\langle x(t), y(t)\rangle$ is a program from $\tilde{y}$ then $\sum_{t=0}^{\infty} A^t c(t)\leqslant \tilde{y}$. Conversely, if $c(t)$ is in R^n_+ for $t\geqslant 0$, and $\tilde{y}$ is in R^n_+, and $\sum_{t=0}^{\infty} A^t c(t)\leqslant \tilde{y}$, then there is a program $\langle x'(t), y'(t)\rangle$ from $\tilde{y}$, with $c'(t)=c(t)$ for $t\geqslant 0$. A program $\langle x(t), y(t)\rangle$ from $\tilde{y}$ is efficient if and only if $\sum_{t=0}^{\infty} A^t c(t)=\tilde{y}$.

Note that the statement of Result 1 is somewhat stronger than that of Theorem 4.1 in Majumdar [9]. A careful reading of Majumdar's proof shows that the stronger statement is fully warranted. We state the result in this stronger form, because we find it most convenient to use it in this form.

Before we proceed to the substantive issues of the following sections, we must establish a preliminary result; viz., an optimal program from strictly positive initial stocks is efficient. This would be obvious if the welfare function was increasing in each component; this need not be the case "at the boundary" in our framework, and it would not be the case when f is of the Cobb–Douglas type.

Result 2. If $\langle \bar{x}(t), \bar{y}(t)\rangle$ is an optimal program from $\tilde{y}$ in R^n_{++}, then $\langle \bar{x}(t), \bar{y}(t)\rangle$ is an efficient program from $\tilde{y}$.

We shall also need the following well-known properties of "supporting prices" in the simple linear production model.

Result 3. Suppose (x^0, y^0) is in Ω, (p^0, q^0) is in $R^n_+\times R^n_+$, and $q^0y^0-p^0x^0\geqslant q^0y-p^0x$ for all (x, y) in Ω. Then the following hold:

(i) $q^0y^0-p^0x^0=0$.

(ii) $q^0-p^0A\leqslant 0$.

(iii) If $y^0\gg 0$ then $q^0=p^0A$.

(iv) If $p^0\gg 0$ then $Ay^0=x^0$.

3. A von Neumann Equilibrium

For any (x, y) in Ω with $x > 0$, let $\lambda(x, y) = \max \{\lambda : y \geqslant \lambda x\}$. It is known (see, for example, Karlin [7, p. 339]) that there is $(\hat{x}, \hat{y})$ in Ω (with $\hat{x} > 0$), $\hat{\lambda} > 0$, and a price vector $\hat{p} > 0$ such that

(i) $\hat{\lambda} = \lambda(\hat{x}, \hat{y}), \hat{y} = \hat{\lambda}\hat{x}$

(ii) $\hat{\lambda} \geqslant \lambda(x, y)$ for all (x, y) in Ω with $x > 0$

(iii) $\hat{p}y \leqslant \hat{\lambda}\hat{p}x$ for all (x, y) in Ω.

We refer to $\hat{x}$ as a vector of *von Neumann stocks*, $\hat{\lambda}$ as the *von Neumann growth factor*, and $\hat{p}$ as a *von Neumann price*. We refer to $(\hat{x}, \hat{y}, \hat{\lambda}, \hat{p})$ as a *von Neumann equilibrium*.

We now relate $\hat{\lambda}$ to the Frobenius eigenvalue of A and $\hat{y}$, $\hat{p}$ to the Frobenius eigenvectors of A.

Since $\hat{x} > 0$, $\hat{\lambda} > 0$, and $\hat{y} = \hat{\lambda}\hat{x}$, so $\hat{y} > 0$. Since $(\hat{x}, \hat{y})$ is in Ω, so $\hat{x} \geqslant A\hat{y}$; and since $\hat{y} > 0$, while $A \gg 0$, so we have $\hat{x} \gg 0$. Since $\hat{y} = \hat{\lambda}\hat{x}$, so $\hat{y} \gg 0$. From (i) and (iii), we have

$$\hat{p}\hat{y} - \hat{\lambda}\hat{p}\hat{x} = 0 \geqslant \hat{p}y - \hat{\lambda}\hat{p}x \qquad \text{for all } (x, y) \text{ in } \Omega. \tag{3.1}$$

Since $\hat{y} \gg 0$, therefore, by Result 3(iii) we have

$$\hat{p} = \hat{\lambda}\hat{p}A, \qquad \text{that is (since } \hat{\lambda} > 0\text{),}$$

$$(1/\hat{\lambda})\,\hat{p} = \hat{p}A. \tag{3.2}$$

Thus, $(1/\hat{\lambda})$ is an eigenvalue of A, and $\hat{p}$ a corresponding eigenvector. Since $\hat{p} > 0$, so $\hat{p}$ is the (left-hand) Frebenius eigenvector of A, and $(1/\hat{\lambda})$ is the (simple) Frobenius eigenvalue of A (Gantmacher [6, p. 63]).

Since $\hat{p} > 0$, and $A \gg 0$, so (3.2) implies that $\hat{p} \gg 0$. Therefore, by Result 3(iv) we get $\hat{x} = A\hat{y}$, so that we get

$$(1/\hat{\lambda})\,\hat{y} = A\hat{y}. \tag{3.3}$$

Since $\hat{y} > 0$, so $\hat{y}$ is the right-hand Frobenius eigenvector of A (Gantmacher [6, p. 63]).

In what follows, we normalize $\hat{y}$, so that $\| \hat{y} \| = 1$. We then normalize $\hat{p}$, so that $\hat{p}\hat{y} = 1$.

We note that since A is productive, [see (A.2) above], there is y^0 such that (Ay^0, y^0) is in Ω, with $y^0 > 0$, and $Ay^0 \ll y^0$. Thus, $\lambda(Ay^0, y^0) > 1$, so that $\hat{\lambda} \geqslant \lambda(Ay^0, y^0) > 1$.

4. Optimality

This section summarizes the main results on optimal growth with discounting in the closed linear model of production.

We start by providing a sufficient condition for the *existence of an optimal program* in our framework. This condition is then maintained for the rest of the discussion. The condition is

(A.4) $\delta\hat{\lambda}^{(1-\alpha)} < 1$.

Discussions of the interpretation of (A.4) are available in Brock and Gale [2] and McFadden [8]. Our existence result can be stated as follows.

Proposition 1. *Given an initial stock $\tilde{y}$ in R^n_+, if $\langle x(t), y(t)\rangle$ is any program from $\tilde{y}$, then*

$$\sum_{t=0}^{\infty} \delta^t w(c(t)) < \infty.$$

Furthermore, there is an optimal program $\langle x^*(t), y^*(t)\rangle$ from $\tilde{y}$.

In view of the existence result above, we can define a *value function*, $V: R^n_+ \to R$, by

$$V(y) = \sum_{t=0}^{\infty} \delta^t w(c^*(t)) \qquad \text{for} \quad y \text{ in } R^n_+,$$

where $\langle x^*(t), y^*(t)\rangle$ is an optimal program from y.

If $\langle x^*(t), y^*(t)\rangle$ is an optimal program from $\tilde{y}$ in R^n_+, then the "principle of optimality" states that for $N \geqslant 0$,

$$V(y^*(0)) = \sum_{t=0}^{N} \delta^t w(c^*(t)) + \delta^{N+1} V(y^*(N+1)).$$

The proof of this principle is too well-known to be reproduced here in detail.

We now note the main results on the price characterization of optimal programs. A competitive program is optimal if it satisfies the transversality condition that the value of input stocks converges to zero (at least for a subsequence of periods).

Proposition 2. *Suppose $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t)\rangle$ is a competitive program from $\tilde{y}$ in R^n_+, and*

$$\liminf_{t\to\infty} \bar{p}(t)\,\bar{x}(t) = 0$$

then $\langle \bar{x}(t), \bar{y}(t)\rangle$ is an optimal program from $\tilde{y}$.

Conversely, an optimal program is competitive and satisfies the transversality condition that the value of input stocks coverges to zero.

PROPOSITION 3. *Suppose* $\langle \bar{x}(t), \bar{y}(t) \rangle$ *is an optimal program from* $\tilde{y}$ *in* R^n_{++}. *Then, there is a sequence* $\langle \bar{p}(t) \rangle$ *such that* $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ *is a competitive program,* $\bar{p}(t) > 0$ *for* $t \geqslant 0$, *and*

$$\lim_{t \to \infty} \bar{p}(t)\, \bar{x}(t) = 0.$$

Remark 2. Our proof of Proposition 3 exploits the simple structure of the production model. It essentially uses the method of Weitzman [11], of obtaining a "price support" for the value function in the *initial* period, but avoids the induction argument for obtaining the "supporting prices" in the subsequent periods for the technology and the value function, by exploiting the structure of the simple polyhedral model. For a more general linear model, a similar result can be proved, following the method of McFadden [8, Theorem 5, pp. 47–48].

It is worth noting that the "prices" $\langle \bar{p}(t) \rangle$ in Proposition 3 also support the value function at $\bar{y}(t)$.

PROPOSITION 4. *Suppose* $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ *is a competitive program from* $\tilde{y}$ *in* R^n_+ *and* $\lim_{t \to \infty} \bar{p}(t)\, \bar{x}(t) = 0$. *Then*

$$\delta^t V(\bar{y})) - \bar{p}(t)\, \bar{y}(t) \geqslant \delta^t V(y) - \bar{p}(t)\, y \qquad \text{for all } y \text{ in } R^n_+,\ t \geqslant 0.$$

COROLLARY. *Suppose* $\langle \bar{x}(t), \bar{y}(t) \rangle$ *is an optimal program from* $\tilde{y}$ *in* R^n_{++}. *Then, there is a sequence* $\langle \bar{p}(t) \rangle$, *such that* $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ *is a competitive program, and*

(i) $\lim_{t \to \infty} \bar{p}(t)\, \bar{x}(t) = 0$

(ii) $\delta^t V(\bar{y}(t)) - \bar{p}(t)\, \bar{y}(t) \geqslant \delta^t V(y) - \bar{p}(t)\, y$ *for all* y *in* R^n_+, $t \geqslant 0$.

Next, we turn to a result on the *existence of a steady-state optimal program*. We show the existence of a stock y^*, such that the program from y^* along which stocks grow at the growth factor

$$g = (\delta \hat{\lambda})^{(1/\alpha)}$$

[that is, along which $y^*(t) = g^t y^*$ for $t \geqslant 0$] is optimal among all programs from y^*.

PROPOSITION 5. *There exists* $y^* \gg 0$, *such that* $y^*(t) = g^t y^*$, $x^*(t) = Ay^*(t+1)$ *for* $t \geqslant 0$ *defines a steady-state optimal program from* y^*, *where*

$$g = (\delta \hat{\lambda})^{(1/\alpha)}.$$

Remark 3. If we define for $\lambda > 0$, $< x(t), y(t), p(t) >$ by $[x(t), y(t), p(t)] = [\lambda x^*(t), \lambda y^*(t), \lambda^{-\alpha} p^*(t)]$ for $t \geqslant 0$, then it follows that $\langle x(t), y(t), p(t) \rangle$ is competitive, and $\lim_{t \to \infty} p(t)\, x(t) = \lim_{t \to \infty} \lambda^{1-\alpha} p^*(t)\, x^*(t) = 0$. Therefore, by Proposition 2, $\langle x(t), y(t) \rangle$ is optimal. Clearly it is also a steady-state program. Hence $\langle x(t), y(t) \rangle$ is a steady-state optimal program. [More generally, it is easily seen by a similar argument that if $\langle \bar{x}(t), \bar{y}(t) \rangle$ is an optimal program from $\bar{y}$, then for any $\lambda > 0$, $\langle \lambda \bar{x}(t), \lambda \bar{y}(t) \rangle$ is an optimal program from $\lambda \bar{y}$.]

The significance of the steady-state optimal program (of Proposition 5) for optimal programs in general is best conveyed through a *turnpike property of optimal programs.* To establish this, we need to strengthen our assumptions on the welfare function. If c and c' are in R^n then by "c is *proportional to* c'," we mean that there is a real number $\mu \neq 0$ such that $c = \mu c'$; otherwise, c *is not proportional to* c'. We now assume, in addition to (A.1)–(A.4), the following property.

(A.5) If c, c' are in R^n_+, c is not proportional to c' and $f(c) > 0 < f(c')$, then, for $0 < \theta < 1$, $f(\theta c + (1-\theta) c') > \theta f(c) + (1-\theta) f(c')$.

PROPOSITION 6. *Suppose* $\langle \bar{x}(t), \bar{y}(t) \rangle$ *is an optimal program from* $\bar{y} \gg 0$. *Then, there is a positive real number* μ, *such that* (i) $(x'(t), y'(t)) = g^t \mu(gAy^*, y^*)$ *for* $t \geqslant 0$ *defines a steady-state optimal program from* μy^*; (ii) $\lim_{t \to \infty} [\bar{y}(t)/g^t] = \mu y^*$ [*where* $y^* \gg 0$ *is the steady-state optimal stock and* g *is the steady-state growth factor obtained in Proposition* 5].

The above turnpike result implies two things about the behavior of stocks along an optimal program $\langle \bar{x}(t), \bar{y}(t) \rangle$. First, the *composition* of the stocks, $[y(t)/\| y(t) \|]$, converges to the *composition* of the steady-state optimal stock, $[y^*/\| y^* \|]$. Second, the *growth factor* of the stocks $[\| y(t+1) \| / \| y(t) \|]$ converges to the *growth factor* of the steady-state optimal program, g.

Since for the optimal program $\langle \bar{x}(t), \bar{y}(t) \rangle$ we will have $\bar{x}(t) = A\bar{y}(t+1)$, so it follows trivially from Proposition 6 that

$$[\bar{x}(t)/g^t] \to \mu x^*(0) \text{ as } t \to \infty \qquad (\text{where } x^*(0) = gAy^*)$$

and

$$[\bar{c}(t)/g^t] \to \mu c^*(0) \text{ as } t \to \infty,$$

where $c^*(0)$ is defined as $[y^* - x^*(0)]$.

5. Decentralization

In this section, we show that optimality of competitive programs can be characterized in terms of the simple decentralizable rule of Brock–Majumdar [3]. However, there is one important qualification. There is a potential choice of the steady–state optimal program (i.e., a choice regarding the appropriate positive scalar multiple) in terms of which the decentralizable rule [(5.1) below] is to be stated. The necessity side of the characterization result (Theorem 2) and Remark 5 show that, for an optimal program from any $\tilde{y}$ in R^n_{++} the decentralizable rule (5.1) is satisfied for *every* steady-state optimal program. Conversely, (as Theorem 1 shows) if $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is a competitive program and the decentralizable rule (5.1) is satisfied with respect to *every* steady-state optimal program $\langle x^*(t), y^*(t) \rangle$ then $\langle \bar{x}(t), \bar{y}(t) \rangle$ is an optimal program. However, this requires verification of the rule, in principle, in infinitely many cases (corresponding to the infinitely many steady-state optimal programs). It is obviously of interest to state the rule in terms of *one particular* steady-state optimal program. A plausible conjecture may be that the rule could be stated in terms of any one of the steady-state optimal programs; in other words, if the rule is satisfied for some steady-state optimal program then (given competitiveness) the program is optimal. If this were true then this would be the simplest possible way to state the rule. In the course of the proof of Theorem 1, i.e., in the course of showing that, if a competitive program is not optimal then there is a steady-state program for which (5.1) is violated, it becomes clear, however, that the steady-state program cannot be arbitrarily chosen. This is confirmed by the example following Theorem 2. It is necessary, therefore, to choose an appropriate steady-state optimal program (given the competitive program), in terms of which, if the decentralizable rule is stated, it signals optimality. Equivalently, given any particular steady-state optimal program, one can define a set of initial stocks (the set Y below), for which the decentralizable rule (for competitive programs from such initial stocks), in terms of the given steady-state program, does signal optimality. We follow this latter approach in Theorem 1.

Denote $\hat{p}\hat{y}/\hat{p}c^*$ by E. Since $c^* > 0$ and $\hat{p} \gg 0$, E is well defined, and since $\hat{y} \gg 0$, $E > 0$. Next, define $\eta = [m(\hat{y})/E \, \| c^* \|]$, and denote

$$\theta = [\eta/(\eta+1)]^{[1/\alpha(1-\alpha)]}.$$

Since $\hat{y} \gg 0$, we have, $\eta > 0$ and $0 < \theta < 1$. Now we define a set of initial stocks for which the exercise will be carried out:

$$Y = \{ y \text{ in } R^n_+ : \hat{p}y = \theta \hat{p} c^* \}.$$

THEOREM 1. *Suppose $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is a competitive program from $\tilde{y}$ in Y. Suppose, for $t \geq 0$,*

$$[\bar{p}(t) - p^*(t)][\bar{y}(t) - y^*(t)] \leq 0 \tag{5.1}$$

then $\langle \bar{x}(t), \bar{y}(t) \rangle$ is optimal from $\tilde{y}$.

Remark 4. Inspecting the rule (5.1), it appears that besides information regarding the "current" commodity prices measured in terms of current utility units $[(1/\delta^t)\,p(t)]$, the output at time t along the competitive program, $[y(t)]$, and $\hat{p}$ and y^*, it is also necessary to know the value of t [for computing $y^*(t)$] in order to be able to verify (5.1). This is an awkward requirement, since the period which is regarded as the origin of measurement of time should not be of any significance and would be an odd information requirement from the point of view of agents at time t. It would seem to be desirable that the rule be in a form where, the information required for its verification is the current prices and current quantities along the competitive program and the information regarding the normalized steady-state compositions, viz., $\hat{p}$, c^* and y^*, and α.

If $\langle p(t)\rangle$ is a sequence of present value prices, let $\langle q(t)\rangle$, defined by $q(t)\equiv(1/\delta^t)\,p(t)$, be the corresponding sequence of current prices. Suppose that we have the current price and quantity information along a competitive program $\langle\bar{x}(t),\ \bar{y}(t),\ \bar{p}(t)\rangle$, viz., $\bar{x}(t)$, $\bar{y}(t)$, and $\bar{q}(t)$. We wish to select $\tilde{q}(t)$ and $\tilde{y}(t)$ for verification of the rule (5.1), rewritten as $[\bar{q}(t)-\tilde{q}(t)][\bar{y}(t)-\tilde{y}(t)]\leqslant 0$ for $t\geqslant 0$. We want this selection to correspond to an optimal steady-state program, which, if used in the verification of (5.1) (in the form just stated), does signal optimality for the given competitive program. Define $\gamma(t)=\frac{1}{2}\bar{q}(t)\,\hat{y}$. Now $\bar{q}(t)\,\hat{y}=(1/\delta^t)\,\bar{p}(t)\,\hat{y}>0$, since $\bar{p}(t)>0$; hence, $\gamma(t)>0$. Denote $(1/\gamma(t)^{1/\alpha})$ by $\beta(t)$ and define $\tilde{c}(t)=\beta(t)\,c^*$, $\tilde{y}(t)=\beta(t)\,y^*$, $\tilde{x}(t)=\beta(t)\,x^*$ (where $x^*=gAy^*$), $\tilde{q}(t)=\gamma(t)\,\hat{p}$. Then it can be shown that $\langle\tilde{x}(t),\ \tilde{y}(t)\rangle$ is an optimal steady-state program with associated present value prices $\tilde{p}(t)=\delta^t\tilde{q}(t)$. Furthermore, it can be checked (essentially by following the method used to prove Theorem 1) that if

$$[\bar{q}(t)-\tilde{q}(t)][\bar{y}(t)-\tilde{y}(t)]\leqslant 0$$

then $\langle\bar{x}(t),\ \bar{y}(t)\rangle$ is an optimal program from $\tilde{y}$.

The above approach, namely that of choosing the appropriate steady-state program, given the competitive program, is equivalent to the approach of Theorem 1. It should be emphasized that, in defining the steady–state values in period t, the value of t was not used. Finally, it may also be remarked that, denoting $v(t)=[\bar{p}(t)-p^*(t)][\bar{y}(t)-y^*(t)]$ for $t\geqslant 0$, it can be checked that $v(t+1)-v(t)\geqslant 0$ for all $t\geqslant 0$ along a competitive program. Hence, the proof of Theorem 1 actually shows that, if the given competitive program is not optimal, then $v(t)>0$ for all but finitely many periods.

A converse of the result of Theorem 1 can now be noted.

THEOREM 2. *Suppose* $\langle\bar{x}(t),\ \bar{y}(t)\rangle$ *is an optimal program from* $\tilde{y}$ *in* Y

and $\tilde{y} \gg 0$. Then, there is a sequence $\langle \bar{p}(t) \rangle$ such that $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is a competitive program, and for $t \geq 0$

$$[\bar{p}(t) - p^*(t)][\bar{y}(t) - y^*(t)] \leq 0.$$

Remark 5. We note that the fact that $\tilde{y}$ is in Y is of no significance in Theorem 2; the result is true without this restriction. However, it is extremely significant in Theorem 1 where the result is not necessarily true without this restriction. We justify this last statement now with an example.

EXAMPLE 1. Let us consider a one-good example, in which $\Omega = \{(x, y)$ in $R_+ \times R_+$: $x \geq ay\}$. Here $0 < a < 1$ ensures that (A.1) and (A.2) are satisfied. The welfare function is $w(c) = c^{1-\alpha}$ (where $0 < \alpha < 1$) for c in R_+; this ensures that (A.3) is satisfied. The discount factor is $0 < \delta < 1$.

It is easy to check that $\hat{y} = 1$, $\hat{x} = a$, $\hat{\lambda} = (1/a)$, $\hat{p} = 1$ is a von Neumann equilibrium, satisfying our stipulated normalizeation of $\hat{y}$ and $\hat{p}$. Assuming $\delta < a^{1-\alpha}$ ensures that (A.4) is satisfied. For the sake of concreteness, choose $a = (\frac{1}{4})$, $\alpha = \frac{1}{2}$, $\delta = \frac{1}{4}$, so that $\hat{\lambda} = 4 = (1/\hat{x})$, and $a^{1-\alpha} = (\frac{1}{4})^{1/2} = \frac{1}{2} > \frac{1}{4} = \delta$.

Define $g = (\delta\hat{\lambda})^{1/\alpha} = 1$, $c^* = (1-\alpha)^{1/\alpha} = \frac{1}{4}$, $y^* = c^*(1 - ga)^{-1} = \frac{1}{3}$. Note then that $\langle x^*(t), y^*(t) \rangle$ defined by $y^*(t) = \frac{1}{3}$, $x^*(t) = \frac{1}{12}$ for $t \geq 0$ is a steady-state program from $y^* = \frac{1}{3}$. Also, $\langle x^*(t), y^*(t), p^*(t) \rangle$ is a competitive program, where $p^*(t) = \frac{1}{4^t}$ for $t \geq 0$, and $p^*(t)\, y^*(t) \to 0$ as $t \to \infty$. So $\langle x^*(t), y^*(t) \rangle$ is an optimal steady-state program.

Next, let $\tilde{y} > 0$, and define $\langle \bar{x}(t), \bar{y}(t) \rangle$ by $\bar{y}(t) = \tilde{y}(\hat{\lambda}^t + g^t)/2 = \tilde{y}(4^t + 1)/2$ for $t \geq 0$, $\bar{x}(t) = \bar{y}(t+1)/\hat{\lambda} = \bar{y}(t+1)/4$ for $t \geq 0$. Then, $\bar{c}(t) = \bar{y}(t) - [\bar{y}(t+1)/\hat{\lambda}] = [\tilde{y}(\hat{\lambda}^t + g^t)/2] - [\tilde{y}(\hat{\lambda}^{t+1} + g^{t+1})/2\hat{\lambda}] = (\tilde{y}/2\hat{\lambda})[\hat{\lambda}^{t+1} + \hat{\lambda}g^t - \hat{\lambda}^{t+1} - g^{t+1}] = (\tilde{y}/2\hat{\lambda})(\hat{\lambda} - g)\, g^t = \frac{3}{8}\tilde{y}$ for $t \geq 0$. Define $\langle \bar{p}(t) \rangle$ as follows: $\bar{p}(t) = \delta^t(1-\alpha)[\bar{c}(t)]^{-\alpha} = \frac{1}{2}(8/3\tilde{y})^{1/2}/4^t$ for $t \geq 0$. One can check that $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is a competitive program. Clearly $\bar{p}(t)\, \bar{y}(t) \to (8/3\tilde{y})^{1/2}\, (\tilde{y}/4)$ as $t \to \infty$, and [since $\langle \bar{p}(t) \rangle$ are the unique competitive prices] $\langle \bar{x}(t), \bar{y}(t) \rangle$ is not an optimal program from $\tilde{y}$.

Now, if we choose $\tilde{y} = 1$, then $\bar{p}(t) = (\frac{2}{3})^{1/2}/4^t < 1/4^t = p^*(t)$ for $t \geq 0$. Also, $\bar{y}(t) = (4^t + 1)/2 > \frac{1}{3} = y^*(t)$ for $t \geq 0$. So, for $t \geq 0$,

$$[\bar{p}(t) - p^*(t)][\bar{y}(t) - y^*(t)] \leq 0$$

but, as we have already noted, $\langle \bar{x}(t), \bar{y}(t) \rangle$ is not optimal from $\tilde{y} = 1$. Thus, the above rule fails to signal the non-optimality of the competitive program, $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$.

The problem is, as we have mentioned earlier in the discussion, that the initial stock $\tilde{y}$ was not chosen carefully enough, given the comparison steady-state program. In this example, $E = 4$, $\eta = 1$, $\theta = (\frac{1}{2})^4 = \frac{1}{16}$, and $Y = \{y : y = \frac{1}{64}\}$. Now, if $\tilde{y}$ is chosen in Y, that is $\tilde{y} = \frac{1}{64}$, then

$\bar{p}(t) = (\frac{128}{3})^{1/2}/4^t$ for $t \geqslant 0$, so that $\bar{p}(t) > p^*(t)$ for $t \geqslant 0$. Also, $\bar{y}(t) = (4^t + 1)/128 > \frac{1}{3} = y^*(t)$ for $t \geqslant 3$. So, for $t \geqslant 3$,

$$[\bar{p}(t) - p^*(t)][\bar{y}(t) - y^*(t)] > 0$$

and the above rule signals the non-optimality of the competitive program $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$.

We note that the definition of the initial stocks (the set Y), given the steady state, does not completely characterize the set of initial stocks for which the decentralizable rule works (given the steady state). The proof of Theorem 1 makes it clear that the essential restriction on the initial stock is that all competitive programs starting from it must satisfy the following inequality:

$$\bar{p}(0)\,\tilde{y} > \hat{p}\hat{y}. \tag{5.2}$$

Initial stocks which are "slightly" larger than those included in the definition of Y may satisfy this. If $\tilde{y} = 1$, $\bar{p}(0)\,\tilde{y} = (\frac{2}{3})^{1/2} < 1 = \hat{p}\hat{y}$, so that inequality (5.2) is violated. If $\tilde{y}$ is chosen to be $\frac{2}{3}$ then $\bar{p}(0)\,\tilde{y} = 1 = \hat{p}\hat{y}$, this being the borderline case where (5.2) is still violated. Here $\bar{p}(t) = 1/4^t = p^*(t)$. Hence $[\bar{p}(t) - p^*(t)][\bar{y}(t) - y^*(t)] = 0$ for $t \geqslant 0$ so that the rule (5.1) still fails to signal non-optimality of $\langle \bar{x}(t), \bar{y}(t) \rangle$. If $\tilde{y} < \frac{2}{3}$, then (5.2) is satisfied and $[\bar{p}(t) - p^*(t)] > 0$ for $t \geqslant 0$. It is clear from the definition of $\bar{y}(t)$ that so long as $\tilde{y} > 0$, $\bar{y}(t) - y^*(t) > 0$ for t sufficiently large and hence (5.1) is violated for t sufficiently large; that is, (5.1) does signal non-optimality. The above discussion illustrates that the critical consideration in the the choice of $\tilde{y}$ (that is, in the definition of Y) is that *all* competitive programs emanating from any $\tilde{y}$ in Y must satisfy (5.2).

6. Proofs

In this section, we provide the proofs of the main results in Sections 2–5 For the detailed proofs of all the results, the reader is referred to the working paper by Dasgupta and Mitra [4].

Proof of Result 2

Let $\langle \bar{x}(t), \bar{y}(t) \rangle$ be an optimal program from $\tilde{y} \gg 0$. Then, there is some time period, s, for which $w(\bar{c}(s)) > 0$. We claim that if $s \geqslant 1$, then $w(\bar{c}(s-1)) > 0$ also. If not, then $w(\bar{c}(s-1)) = 0$. Choose $0 < \lambda < 1$, with λ sufficiently close to 1, so that

$$[w(A\bar{c}(s))/(1-\lambda)^{\alpha}] \geqslant \delta w(\bar{c}(s)). \tag{6.1}$$

Note that $\bar{c}(s)>0$, so $A\bar{c}(s)\gg 0$, and so $w(A\bar{c}(s))>0$, so that by suitable choice of λ, the inequality (6.1) can be satisfied.

Consider a sequence $\langle x'(t), y'(t)\rangle$ defined by $[x'(t), y'(t)]=[\bar{x}(t), \bar{y}(t)]$ for $t\neq s$, $s-1$; $y'(s-1)=\bar{y}(s-1)$, $y'(s)=[\lambda\bar{c}(s)+\bar{x}(s)]$; $x'(s-1)=Ay'(s)$, $x'(s)=\bar{x}(s)$. Note that $y'(s)\geqslant x'(s)$, and $x'(s-1)=Ay'(s)\leqslant A\bar{y}(s)\leqslant \bar{x}(s-1)\leqslant\bar{y}(s-1)=y'(s-1)$. Hence, $\langle x'(t), y'(t)\rangle$ is a program from y. Also, $c'(t)=\bar{c}(t)$ for $t\neq s$, $s-1$; $c'(s)=\lambda\bar{c}(s)$, and $c'(s-1)=y'(s-1)-x'(s-1) = \bar{y}(s-1)-Ay'(s) = \bar{y}(s-1)-A\bar{x}(s)-\lambda A\bar{c}(s)-(1-\lambda)A\bar{c}(s)+(1-\lambda)A\bar{c}(s) = \bar{y}(s-1)-A[\bar{x}(s)+\bar{c}(s)]+(1-\lambda)A\bar{c}(s) = \bar{y}(s-1)-A\bar{y}(s)+(1-\lambda)A\bar{c}(s)\geqslant\bar{c}(s-1)+(1-\lambda)A\bar{c}(s)\geqslant(1-\lambda)A\bar{c}(s)$.

Then, $w(c'(s-1))+\delta w(c'(s))\geqslant w((1-\lambda)A\bar{c}(s))+\delta w(\lambda\bar{c}(s))=(1-\lambda)^{1-\alpha} w(A\bar{c}(s))+\delta\lambda^{1-\alpha}w(\bar{c}(s)) > (1-\lambda)[w(A\bar{c}(s))/(1-\lambda)^{\alpha}]+\delta\lambda\, w(\bar{c}(s))$ [since $\lambda^{1-\alpha}>\lambda$ and $w(\bar{c}(s))>0]\geqslant(1-\lambda)\delta w(\bar{c}(s))+\delta\lambda w(\bar{c}(s))$ [using (6.1)] $=\delta w(\bar{c}(s))=w(\bar{c}(s-1))+\delta w(\bar{c}(s))$. This shows that [since $c'(t)=\bar{c}(t)$ for $t\neq s$, $s-1$] $\langle\bar{x}(t), \bar{y}(t)\rangle$ is not optimal, a contradiction which proves our claim.

In view of this, repeating the above argument for a finite number of periods, we can conclude that $w(\bar{c}(0))>0$.

Now, we claim that $\langle\bar{x}(t), \bar{y}(t)\rangle$ is efficient. Otherwise, by Result 1,

$$\sum_{t=0}^{\infty} A^t\bar{c}(t)<\tilde{y}.$$

Defining $c(0)=\bar{c}(0)+[\tilde{y}-\sum_{t=0}^{\infty}A^t\bar{c}(t)]>\bar{c}(0)$, and $c(t)=\bar{c}(t)$ for $t\geqslant 1$, we note that

$$\sum_{t=0}^{\infty} A^t c(t)=\tilde{y}.$$

So, by Result 1, there is a program $\langle x'(t), y'(t)\rangle$ from $\tilde{y}$ with $c'(t)=c(t)$ for $t\geqslant 0$. Now, $w(c'(t))=w(c(t))=w(\bar{c}(t))$ for $t\geqslant 1$, and $w(c'(0))=w(c(0))>w(\bar{c}(0))$, using (A.3) and the facts that $c(0)>\bar{c}(0)$ and $w(\bar{c}(0))>0$ [and so $f(\bar{c}(0))>0$]. This shows that $\langle\bar{x}(t), \bar{y}(t)\rangle$ is not optimal from $\tilde{y}$, a contradiction. Hence $\langle\bar{x}(t), \bar{y}(t)\rangle$ is efficient. ▌

Proof of Proposition 1

Given an initial stock, $\tilde{y}$ in R^n_+, define $B=[1/m(\hat{p})]$, $\hat{B}=[w(B\hat{p}\tilde{y}e)/(1-\delta\hat{\lambda}^{1-\alpha})]$ and a sequence $\langle k(t)\rangle$ by $k(t)=\hat{\lambda}^t(B\hat{p}\tilde{y})\,e$ for $t\geqslant 0$. Note that $\sum_{t=0}^{\infty}\delta^t w(k(t))$ is a convergent geometric series [given (A.4)] and clearly

$$\sum_{t=0}^{\infty}\delta^t w(k(t))=\hat{B}.$$

Next, let $\langle x(t), y(t)\rangle$ be a program from $\tilde{y}$. Then, for $t \geqslant 0$,

$$\hat{p}y(t+1) = \hat{\lambda}\hat{p}Ay(t+1) \leqslant \hat{\lambda}\hat{p}x(t) \leqslant \hat{\lambda}\hat{p}y(t)$$

so that $\hat{p}y(t) \leqslant \hat{\lambda}^t\hat{p}\tilde{y}$ for all $t \geqslant 0$. Thus, for $i = 1, ..., n$, $y_i(t) \leqslant \hat{\lambda}^t B\hat{p}\tilde{y}$ for $t \geqslant 0$, and so $y(t) \leqslant \hat{\lambda}^t(B\hat{p}\tilde{y})\, e \equiv k(t)$ for $t \geqslant 0$. Since $c(t) \leqslant y(t)$, so $c(t) \leqslant k(t)$ and $w(c(t)) \leqslant w(k(t))$ for $t \geqslant 0$.

It is, of course, clear that $\sum_{t=0}^{T} \delta^t w(c(t))$ is bounded above by $\hat{B}$ and is monotonically non-decreasing in T, so it converges, and $\sum_{t=0}^{\infty} \delta^t w(c(t)) \leqslant \hat{B}$.

Let $\langle x(t), y(t)\rangle$ be any program from $\tilde{y}$. Then

$$\sum_{t=0}^{\infty} \delta^t[w(k(t)) - w(c(t))] \leqslant \sum_{t=0}^{\infty} \delta^t w(k(t)) = \hat{B}.$$

So, by using Brock–Gale [2, Lemma 2, p. 236], there is a program $\langle x^*(t), y^*(t)\rangle$ from $\tilde{y}$, such that

$$\sum_{t=0}^{\infty} \delta^t[w(k(t)) - w(c^*(t))] \leqslant \sum_{t=0}^{\infty} \delta^t[w(k(t)) - w(c(t))]$$

for every program $\langle x(t), y(t)\rangle$ from $\tilde{y}$. This means

$$\sum_{t=0}^{\infty} \delta^t w(c^*(t)) \geqslant \sum_{t=0}^{\infty} \delta^t w(c(t))$$

for every program $\langle x(t), y(t)\rangle$ from $\tilde{y}$. Hence $\langle x^*(t), y^*(t)\rangle$ is an optimal program from $\tilde{y}$. ∎

Proof of Proposition 3

Define the sets G and H as follows:

$$G = \left\{(a, b) \text{ in } R \times R^n\colon a \leqslant \sum_{t=0}^{\infty} \delta^t w(c(t)) - \sum_{t=0}^{\infty} \delta^t w(\bar{c}(t)),\right.$$
$$b \leqslant \sum_{t=0}^{\infty} A^t\bar{c}(t) - \sum_{t=0}^{\infty} A^t c(t),$$
$$\left.\text{for some program } \langle x(t), y(t)\rangle\right\}. \tag{6.2}$$

It is worth emphasizing that the program referred to in the definition of G, $\langle x(t), y(t)\rangle$, need not satisfy $y(0) = \bar{y}(0)$.

$$H = \{(a, b) \text{ in } R \times R^n\colon a > 0, b \gg 0\}. \tag{6.3}$$

Clearly, G and H are non-empty, convex sets, and H has a non-empty interior. Also, G and H are disjoint. For if (a, b) belongs to both G and H, then there is some program $\langle x(t), y(t)\rangle$ such that

$$\sum_{t=0}^{\infty} A^t c(t) \leqslant \sum_{t=0}^{\infty} A^t \bar{c}(t) \tag{6.4}$$

and

$$\sum_{t=0}^{\infty} \delta^t w(c(t)) > \sum_{t=0}^{\infty} \delta^t w(\bar{c}(t)). \tag{6.5}$$

Since $\langle \bar{x}(t), \bar{y}(t)\rangle$ is optimal, it is efficient. Hence, by Result 1, $\sum_{t=0}^{\infty} A^t \bar{c}(t) = \tilde{y}$, and (6.4) implies that $\sum_{t=0}^{\infty} A^t c(t) \leqslant \tilde{y}$. So, by Result 1, there is a program $\langle x'(t), y'(t)\rangle$ from $\tilde{y}$, with $c'(t) = c(t)$ for $t \geqslant 0$. In view of (6.5), $\langle \bar{x}(t), \bar{y}(t)\rangle$ is then not optimal, a contradiction. This proves that G and H are disjoint.

Using Theorem 3.5 of Nikaido [10, p. 35], we have (Q, P) in $R_+ \times R^n_+$ such that

$$Qa + Pb \leqslant 0 \qquad \text{for all } (a, b) \text{ in } G. \tag{6.6}$$

We claim now that $Q > 0$. If not, then $Q = 0$, and $P > 0$, and (6.6) implies that

$$Pb \leqslant 0 \qquad \text{for all } (a, b) \text{ in } G. \tag{6.7}$$

Define $\langle x(t), y(t)\rangle$ by $y(t) = \frac{1}{2}\bar{y}(t), x(t) = Ay(t+1)$ for $t \geqslant 0$. Then $\langle x(t), y(t)\rangle$ is a program from $[\tilde{y}/2]$. Now, by Result 1, $\sum_{t=0}^{\infty} A^t \bar{c}(t) = \tilde{y}$, and $\tilde{y} \gg 0$. So, choosing $b = \sum_{t=0}^{\infty} A^t \bar{c}(t) - \sum_{t=0}^{\infty} A^t c(t) \geqslant [\tilde{y}/2] \gg 0$ and $a = \sum_{t=0}^{\infty} \delta^t w(c(t)) - \sum_{t=0}^{\infty} \delta^t w(\bar{c}(t))$, we note that (a, b) is in G, and $Pb > 0$, contradicting (6.7). Hence $Q > 0$. Define $p = P/Q$, and note that $p \geqslant 0$; also, using (6.6)

$$a + pb \leqslant 0 \qquad \text{for all } (a, b) \text{ in } G. \tag{6.8}$$

Thus, given any program $\langle x(t), y(t)\rangle$, we have

$$\sum_{t=0}^{\infty} \delta^t w(c(t)) - p\sum_{t=0}^{\infty} A^t c(t) \leqslant \sum_{t=0}^{\infty} \delta^t w(\bar{c}(t)) - p\sum_{t=0}^{\infty} A^t \bar{c}(t). \tag{6.9}$$

Now, given any c in R^n_+, and $s \geqslant 0$, define $c(t) = \bar{c}(t)$ for $t \neq s$, $c(t) = c$ for $t = s$. Then, there is a program $\langle x'(t), \bar{y}(t)\rangle$ from $\sum_{t=0}^{\infty} A^t c(t)$, with $c'(t) = c(t)$ for $t \geqslant 0$, by Result 1. Using this in (6.9), we obtain

$$\delta^s w(c(s)) - pA^s c \leqslant \delta^s w(\bar{c}(s)) - pA^s \bar{c}(s). \tag{6.10}$$

Define $\bar{p}(t)=pA^t$ for $t \geqslant 0$. Then (6.10) implies

$$\delta^t w(\bar{c}(t))-\bar{p}(t)\,\bar{c}(t) \geqslant \delta^t w(c)-\bar{p}(t)\,c \qquad \text{for all } c \text{ in } R^n_+. \tag{6.11}$$

Also, for $t \geqslant 0$, we have $\bar{p}(t+1)\,\bar{y}(t+1)-\bar{p}(t)\,\bar{x}(t)=\bar{p}(t+1)\,\bar{y}(t+1)-\bar{p}(t)\,A\bar{y}(t+1)$ [since $\bar{x}(t)=A\bar{y}(t+1)$ by efficiency of $\langle \bar{x}(t), \bar{y}(t)\rangle$] $=0$. And, for $t \geqslant 0$, and any (x, y) in Ω, $\bar{p}(t+1)\,y-\bar{p}(t)\,x \leqslant \bar{p}(t+1)\,y-\bar{p}(t)\,Ay=0$. Thus, for $t \geqslant 0$,

$$\bar{p}(t+1)\,\bar{y}(t+1)-\bar{p}(t)\,\bar{x}(t) \geqslant \bar{p}(t+1)\,y-\bar{p}(t)\,x \qquad \text{for all } (x, y) \text{ in } \Omega. \tag{6.12}$$

Clearly (6.11), (6.12) show that $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t)\rangle$ is a competitive program.

Finally, note that for $T \geqslant 1$,

$$\begin{aligned} p\left[\sum_{t=0}^{T} A^t\bar{c}(t)\right] &= p\left[\sum_{t=0}^{T} A^t[\bar{y}(t)-\bar{x}(t)]\right] \\ &= p\left[\bar{y}(0)+\sum_{t=0}^{T-1} A^{t+1}\bar{y}(t+1)-\sum_{t=0}^{T-1} A^t\bar{x}(t)-A^T\bar{x}(T)\right] \\ &= p[\bar{y}(0)-A^T\bar{x}(T)]. \end{aligned}$$

Hence,

$$\begin{aligned} p\sum_{t=0}^{\infty} A^t\bar{c}(t) &= p\bar{y}(0)-\lim_{t\to\infty} pA^t\bar{x}(t) \\ &= p\bar{y}(0)-\lim_{t\to\infty} \bar{p}(t)\,\bar{x}(t). \end{aligned}$$

Since $\sum_{t=0}^{\infty} A^t\bar{c}(t)=\bar{y}(0)$ by efficiency, so $p\sum_{t=0}^{\infty} A^t\bar{c}(t)=p\bar{y}(0)$. Hence $\lim_{t\to\infty} \bar{p}(t)\,\bar{x}(t)=0$, which proves the proposition. ∎

Remark. It is worth noting that the price sequence $\langle \bar{p}(t)\rangle$ will, in fact, satisfy $\bar{p}(t) \gg 0$ for $t \geqslant 0$ (see Lemma 2 below). So the price, p, obtained from the separation theorem must satisfy $p > 0$.

The proof of Proposition 5 (the existence of a steady-state optimal program) requires a sequence of preliminary results which we now discuss. First, using the assumption on the utility function [(A.3)], it is easy to prove the following result.

LEMMA 1. (a) *Suppose p^0 is in R^n_+. Then there exists $\theta^0 > 0$ such that*

$$w(\theta e)-p^0(\theta e) > 0 \qquad \textit{for} \quad 0 < \theta \leqslant \theta^0. \tag{6.13}$$

(b) *Suppose* p^0 *and* c^0 *are in* R^n_+, *respectively. If* $w(c^0)-p^0c^0 \geqslant w(c)-p^0c$ *for all* c *in* R^n_+, *then* $w(c^0)-p^0c^0>0$, $w(c^0)>0$, $c^0>0$.

Then, using Lemma 1, one can establish that competitive programs are "interior" in input–output levels.

LEMMA 2. *Suppose* $\langle x(t), y(t), p(t)\rangle$ *is a competitive program from* $\tilde{y}$ *in* R^n_+. *Then* (i) $w(c(t))>0$, $c(t)>0$, $x(t))\gg 0$, $y(t)\gg 0$, *and* $p(t)\gg 0$; (ii) $p(t+1)=p(t)\,A$ *for* $t\geqslant 0$ *and* $Ay(t+1)=x(t)$ *for* $t\geqslant 0$.

Proof. The only non-trivial part of (i) is to show that $p(t)\gg 0$ for $t\geqslant 0$. Using (2.1) we have for $t\geqslant 0$, $i=1,\ldots,n$, $\delta^t w(c(t))-p(t)\,c(t)\geqslant \delta^t w(c(t)+e^i)-p(t)(c(t)+e^i)$. Hence, for $i=1,\ldots,n$, $t\geqslant 0$, $p_i(t)=p(t)\,e^i\geqslant \delta^t[w(c(t)+e^i)-w(c(t))]$. Since $w(c(t))>0$, we have $f(c(t))>0$ and by (A.3) (c), $f(c(t)+e^i)>f(c(t))$. Therefore, $w(c(t)+e^i)>w(c(t))$, and $p_i(t)>0$.

Part (ii) follows from Result 3(iii) and 3(iv), and (2.2), since $y(t)\gg 0$ and $p(t)\gg 0$ for $t\geqslant 0$ by part (i) above. ▮

Lemma 2 in turn yields the result that positive scalar multiples of competitive programs are also competitive programs.

LEMMA 3. (a) *If* c^0 *in* R^n_+ *and* p^0 *in* R^n_+ *are such that,* $w(c^0)-p^0c^0\geqslant w(c))-p^0c$ *for all* c *in* R^n_+, *then for any* $\beta>0$, $w(\hat{\beta}c^0)-\beta p^0(\hat{\beta}c^0)\geqslant w(c)-\beta p^0c$ *for all* c *in* R^n_+, *where* $\hat{\beta}\equiv[1/\beta^{1/\alpha}]$.

(b) *If* $\langle x(t), y(t), p(t)\rangle$ *is a competitive progam from* $\tilde{y}$ *with consumption sequence* $\langle c(t)\rangle$, *then for any* $\beta>0$, $\langle \beta x(t), \beta y(t), \beta^{-\alpha}p(t)\rangle$ *is a competitive program from* $\beta\tilde{y}$, *with consumption sequence* $\langle \beta c(t)\rangle$.

Proof. Suppose c^0 and p^0 are in R^n_+ and $w(c^0)-p^0c^0\geqslant w(c)-p^0c$ for all c in R^n_+. Consider any $\beta>0$ and any c in R^n_+. Then

$$\begin{aligned} w(\hat{\beta}c^0)-\beta p^0(\hat{\beta}c^0) &= \beta\hat{\beta}[w(c^0)-p^0c^0] \\ &\geqslant \beta\hat{\beta}[w(c/\hat{\beta})-p^0(c/\hat{\beta})]=w(c)-\beta p^0c. \end{aligned}$$

This proves (a). To prove (b), note that by Lemma 2, $x(t)=Ay(t+1)$ for $t\geqslant 0$. Hence, $\beta x(t)=A\beta y(t+1)$. Since $0\leqslant c(t)=y(t)-x(t)$ for $t\geqslant 0$, therefore, $0\leqslant \beta c(t)=\beta y(t)-\beta x(t)$ for $t\geqslant 0$. Hence, $\langle \beta x(t), \beta y(t)\rangle$ is a program with consumption sequence $\langle \beta c(t)\rangle$. To see that $\langle \beta x(t), \beta y(t), \beta^{-\alpha}p(t)\rangle$ is competitive, we note that clearly $\beta^{-\alpha}p(t)\geqslant 0$, and $w(\beta c(t))-\beta^{-\alpha}p(t)(\beta c(t))\geqslant w(c)-\beta^{-\alpha}p(t)\,c$ for any c in R^n_+, by part (a) above. We need to show that $\beta^{-\alpha}p(t+1)\ (\beta y(t+1))-\beta^{-\alpha}p(t)(\beta x(t))\geqslant$

$\beta^{-\alpha}p(t+1)\,y-\beta^{-\alpha}p(t)\,x$ for any (x, y) in Ω, $t \geqslant 0$. This follows from (2.2); that is,

$$p(t+1)\,y(t+1)-p(t)\,x(t) \geqslant p(t+1)\,y-p(t)\,x \qquad \text{for all } (x, y) \text{ in } \Omega,\ t \geqslant 0,$$

which in turn implies that for any (x, y) in Ω, and $t \geqslant 0$,

$$\begin{aligned}
&\beta^{-\alpha}p(t+1)\,\beta y(t+1)-\beta^{-\alpha}p(t)\,\beta x(t)\\
&\quad \geqslant \beta^{-\alpha}p(t+1)\,\beta(y/\beta)-\beta^{-\alpha}p(t)\,\beta(x/\beta)\\
&\qquad \left[\text{since } (x, y) \text{ is in } \Omega \text{ implies } \left(\frac{1}{\beta}x, \frac{1}{\beta}y\right) \text{ is in } \Omega\right]\\
&\quad = \beta^{-\alpha}p(t+1)\,y-\beta^{-\alpha}p(t)\,x
\end{aligned}$$

which completes the proof. ▮

Using Lemma 3 now yields the existence result on a steady-state optimal program (Proposition 5) which we now prove.

Proof of Proposition 5

We first define a set

$$S=\{c \text{ in } R^n_+ : \|c\| \leqslant [w(e)/m(\hat{p})]^{1/\alpha}\}.$$

We note that for c in R^n_+, c not in S, we have

$$w(c)-\hat{p}c<0. \tag{6.14}$$

To see this note that for c not in S,

$$\begin{aligned}
w(c)-\hat{p}c &= \|c\|[\{w(c)/\|c\|\}-\{\hat{p}c/\|c\|\}]\\
&\leqslant \|c\|[\{w(c/\|c\|)/\|c\|^{\alpha}\}-m(\hat{p})]\\
&\leqslant \|c\|[w(e)/\|c\|^{\alpha}-m(\hat{p})]\\
&= \|c\|^{1-\alpha}\,m(\hat{p})[w(e)/m(\hat{p})-\|c\|^{\alpha}]\\
&<0, \qquad \text{since } \|c\|>[w(e)/m(\hat{p})]^{1/\alpha}.
\end{aligned}$$

We also note that, by Lemma 1, there exists θ' such that

$$w(\theta' e)-\hat{p}(\theta' e)>0 \qquad \text{and} \qquad 0<\theta'<[w(e)/\|\hat{p}\|]^{1/\alpha}/n. \tag{6.15}$$

Now, S is a non-empty, compact set in R^n_+, and the function, $F(c) \equiv w(c)-\hat{p}c$ for c in R^n_+, is continuous on S. So, there is c^* in S, such that

$$w(c^*)-\hat{p}c^* \geqslant w(c)-\hat{p}c \qquad \text{for all } c \text{ in } S. \tag{6.16}$$

Let $c' = \theta' e$. Then $\| c' \| = n\theta' < [w(e)/\| \hat{p} \|]^{1/\alpha} \leqslant [w(e)/m(\hat{p})]^{1/\alpha}$, so c' is in S. Using this in (6.16), we have $w(c^*) - \hat{p}c^* \geqslant w(c') - \hat{p}c' > 0$ [by (6.15)]. This also implies $c^* > 0$, by Remark 1 in Section 2b. Since for c not in S, $w(c) - \hat{p}c < 0$, so by (6.16)

$$w(c^*) - \hat{p}c^* \geqslant w(c) - \hat{p}c \qquad \text{for all } c \text{ in } R^n_+. \tag{6.17}$$

Define $g = (\delta\hat{\lambda})^{1/\alpha}$, and note that by (A.4), we have

$$g < \left[\left(\frac{1}{\hat{\lambda}^{1-\alpha}}\right)\hat{\lambda}\right]^{1/\alpha} = [\hat{\lambda}^\alpha]^{1/\alpha} = \hat{\lambda} \tag{6.18}$$

so that, by the Frobenius theorem, we have $(I - gA)$ is non-singular (invertible), and $(I - gA)^{-1} \gg 0$ (Nikaido [10, p. 102 and p. 107]). Define

$$y^* = c^*(I - gA)^{-1}. \tag{6.19}$$

Now, define the sequence $\langle x^*(t), y^*(t) \rangle$ from y^* by $y^*(0) = y^*$, $y^*(t+1) = gy^*(t)$ for $t \geqslant 0$; $x^*(t) = Ay^*(t+1)$ for $t \geqslant 0$. Note, then, that $(x^*(t), y^*(t+1))$ is in Ω for $t \geqslant 0$. Also $y^*(t) + x^*(t) = y^*(t) - Ay^*(t+1) = g^t y^* - Ag^{t+1}y^* = g^t[I - gA]y^* = c^*g^t$ [by (6.19)] $\geqslant 0$. Hence $\langle x^*(t), y^*(t) \rangle$ is a program from y^*, and $c^*(t) = c^*g^t$ for $t \geqslant 0$ is the corresponding consumption sequence. Note that $c^* > 0$, $(I - gA)^{-1} \gg 0$ implies that $y^* \gg 0$ and hence $x^* \gg 0$.

Clearly, $\langle x^*(t), y^*(t) \rangle$ is a steady-state program. Our next task is to show that it is also an optimal program. To this end, define

$$p^*(t) = [\hat{p}/\hat{\lambda}^t] \qquad \text{for} \quad t \geqslant 0. \tag{6.20}$$

We will now show that for $t \geqslant 0$,

(i) $\delta^t w(c^*(t)) - p^*(t)\, c^*(t) \geqslant \delta^t w(c) - p^*(t)\, c$ for all c in R^n_+

(ii) $0 = p^*(t+1)\, y^*(t+1) - p^*(t)\, x^*(t) \geqslant p^*(t+1)\, y - p^*(t)\, x$ for all (x, y) in Ω.

To establish (i), note that by (6.17) and Lemma 3(a) we have

$$w(g^t c^*) - (1/(g^t)^\alpha)\, \hat{p}(g^t c^*) \geqslant w(c) - (1/(g^t)^\alpha)\, \hat{p}c \qquad \text{for all } c \text{ in } R^n_+,\ t \geqslant 0.$$

Therefore, using $g^\alpha = \delta\hat{\lambda}$ and $c^*(t) = g^t c^*$ for $t \geqslant 0$,

$$w(c^*(t)) - (1/\delta^t\hat{\lambda}^t)\, \hat{p}c^*(t) \geqslant w(c) - (1/\delta^t\hat{\lambda}^t)\, \hat{p}c \qquad \text{for all } c \text{ in } R^n_+,\ t \geqslant 0,$$

which is (i).

To establish (ii) note that for $t \geqslant 0$, $p^*(t+1) - p^*(t)A = (1/\hat{\lambda}^{t+1})(\hat{p} - \lambda\hat{p}A) = 0$. Therefore, for any (x, y) in Ω and $t \geqslant 0$ we have

$$\begin{aligned} p^*(t+1)\,y - p^*(t)\,x &\leqslant p^*(t+1)\,y - p^*(t)\,Ay = 0 \\ &= p^*(t+1)\,y^*(t+1) - p^*(t)\,Ay^*(t+1) \\ &= p^*(t+1)\,y^*(t+1) - p^*(t)\,x^*(t). \end{aligned}$$

Finally, note that $p^*(t)\,y^*(t) = (\hat{p}/\hat{\lambda}^t)\ \ y^*g^t = \hat{p}y^*\,(g/\hat{\lambda})^t$. Since $0 < (g/\hat{\lambda}) < 1$ [by (6.18)], so $p^*(t)\,y^*(t) \to 0$ as $t \to \infty$; that is, the transversality condition is satisfied. Hence $\langle x^*(t),\ y^*(t)\rangle$ is optimal from y^* by Proposition 1. Since it is a steady-state program, so it is a steady-state optimal program. ▌

The proof of Proposition 6 (the turnpike property of optimal programs) requires several preliminary results, which we now discuss. First, using Lemma 3 above, it is easy to prove the following result.

LEMMA 4. *Suppose* $\langle x(t), y(t), p(t)\rangle$ *is a competitive program from* $\tilde{y}$ *in* R^n_+. *Then*

(i) $w(c(t)/g^t) - \hat{\lambda}^t p(t)(c(t)/g^t) \geqslant w(c) - \hat{\lambda}^t p(t)\,c$ *for all* c *in* R^n_+, $t \geqslant 0$; *and*

(ii) $w(c(t)/g^t) - \hat{\lambda}^t p(t)(c(t)/g^t) > 0$ *for* $t \geqslant 0$.

The additional assumption on the welfare function [(A.5)] effectively makes it strictly concave everywhere in the part of the domain of w which is of interest, namely where $w(c) > 0$. This is what the following Lemma establishes.

LEMMA 5. *Under* (A.3) *and* (A.5), *if* c^0, c^1 *are in* R^n_+, $c^0 \neq c^1$, $w(c^1) > 0 < w(c^0)$, *and* $0 < \theta < 1$, *then*

$$w(\theta c^0 + (1-\theta)\,c^1) > \theta w(c^0) + (1-\theta)\,w(c^1).$$

Proof. If c^0 is not proportional to c^1, then,

$$\begin{aligned} &w(\theta c^0 + (1-\theta)\,c^1) \\ &\quad = [\,f(\theta c^0 + (1-\theta)\,c^1)]^{1-\alpha} \\ &\quad \geqslant [\theta f(c^0) + (1-\theta) f(c^1)]^{1-\alpha} \qquad \text{[by (A.5)]} \\ &\quad \geqslant \theta(\,f(c^0))^{1-\alpha} + (1-\theta)(f(c^1))^{1-\alpha} \\ &\qquad \text{[since } x^{1-\alpha} \text{ is concave for } x \geqslant 0] \\ &\quad = \theta w(c^0) + (1-\theta)\,w(c^1). \end{aligned}$$

If c^0 is proportional to c^1, then there exists $\mu \neq 0$ such that $c^0 = \mu c^1$. Since c^0, c^1 are in R^n_+, therefore, $\mu > 0$. Since $c^0 \neq c^1$, therefore, $\mu \neq 1$. Therefore,

$$\begin{aligned}
& w(\theta c^0 + (1-\theta)\, c^1) \\
&= w((\theta\mu + 1 - \theta)\, c^1) \\
&= (\theta\mu + 1 - \theta)^{1-\alpha}\, w(c_1) \\
&> [\theta\mu^{1-\alpha} + (1-\theta)\, 1^{1-\alpha}]\, w(c_1) \\
&\quad [\text{since, } x^{1-\alpha} \text{ is a strictly concave function of } x > 0, \\
&\quad \mu \neq 1, \mu > 0 \text{ and } 0 < \theta < 1] \\
&= \theta\mu^{1-\alpha} w(c^1) + (1-\theta)\, w(c^1) \\
&= \theta w(\mu c^1) + (1-\theta)\, w(c^1) \\
&= \theta w(c^0) + (1-\theta)\, w(c^1).
\end{aligned}$$

This establishes the lemma. ∎

Since the welfare function is strictly concave in the relevant domain, therefore, convergence of the *supporting prices* of a sequence of consumption points to that of a given point implies the convergence of the sequence of consumption points to the given point. This is the content of the following lemma.

LEMMA 6. *Under* (A.3) *and* (A.5), *if* $\langle p^s \rangle$, $\langle c^s \rangle$ *are sequences in* R^n_+, $\bar{p}$ *and* $\bar{c}$ *are in* R^n_+ *and*

(i) $\lim_{s \to \infty} p^s = \bar{p}$,

(ii) $w(c^s) - p^s c^s \geqslant w(c) - p^s c$ *for all* c *in* R^n_+, *for each* $s \geqslant 0$,

(iii) $w(\bar{c}) - \bar{p}\bar{c} \geqslant w(c) - \bar{p}c$ *for all* c *in* R^n_+,

then $\lim_{s \to \infty} c^s = \bar{c}$.

Proof. First note that, by Lemma 1(b), (ii) and (iii), respectively, imply that

$$w(c^s) > 0 \quad \text{for} \quad s \geqslant 0; \qquad \text{and} \qquad w(\bar{c}) > 0. \tag{6.21}$$

Now, suppose the lemma is false. Then, without any loss of generality, we may suppose that there exists $\varepsilon_0 > 0$ such that

$$\| c^s - \bar{c} \| > \varepsilon_0 \qquad \text{for} \quad s \geqslant 0 \tag{6.22}$$

and [by virtue of the continuity of w and (6.21) above] that

$$w(c) > 0, \qquad \text{for } c \text{ in } R^n_+ \text{ satisfying } \| c - \bar{c} \| \leqslant \varepsilon_0. \tag{6.23}$$

Let $B = \{c \text{ in } R^n_+ : \|c - \bar{c}\| = \varepsilon_0\}$. Clearly B is compact. Consider any c in B. Then by Lemma 5, we have

$$w((\bar{c}/2) + (c/2)) > \tfrac{1}{2}(w(\bar{c}) + w(c))$$

[since $c \neq \bar{c}$ and $w(c) > 0$, $0 < w(\bar{c})$ and, therefore, Lemma 5 applies]. Therefore, $2[w((\bar{c}/2) + (c/2)) - w(\bar{c})] - [w(c) - w(\bar{c})] > 0$ for c in B. Since B is compact and w is continuous, therefore, there exists $\varepsilon_1 > 0$ such that

$$2[w((\bar{c}/2) + (c/2)) - w(\bar{c})] - [w(c) - w(\bar{c})] \geqslant \varepsilon_1 \qquad \text{for } c \text{ in } B. \quad (6.24)$$

Now, consider any $s \geqslant 0$. Since $\|c^s - \bar{c}\| > \varepsilon_0$, therefore, there exist $\lambda^1 > 1$ and c^1 in B such that $(1/\lambda^1)\, c^s + (1 - (1/\lambda^1))\, \bar{c} = c^1$. Since $w(c^s) > 0$, $w(\bar{c}) > 0$, $c^s \neq \bar{c}$, and $0 < (1/\lambda^1) < 1$, therefore, by Lemma 5, $w(c^1) > (1/\lambda^1)\, w(c^s) + (1 - (1/\lambda^1))\, w(\bar{c})$. Since $\lambda^1 > 0$, therefore, $\lambda^1[w(c^1) - w(\bar{c})] > w(c^s) - w(\bar{c})$. Therefore,

$$(\lambda^1 - 1)[w(c^1) - w(\bar{c})] > w(c^s) - w(c^1). \qquad (6.25)$$

Now, from (ii), substituting c^1 for c, we obtain

$$\begin{aligned} w(c^1) - p^s c^1 &\leqslant w(c^s) - p^s c^s = w(c^s) - p^s[\lambda^1 c^1 + (1 - \lambda^1)\, \bar{c}] \\ &= w(c^s) - p^s \bar{c} - \lambda^1 p^s (c^1 - \bar{c}). \end{aligned}$$

Therefore,

$$(\lambda^1 - 1)\, p^s (c^1 - \bar{c}) \leqslant w(c^s) - w(c^1) < (\lambda^1 - 1)[w(c^1) - w(\bar{c})] \qquad \text{[from (6.25)]}.$$

Since $\lambda^1 - 1 > 0$, therefore,

$$p^s (c^1 - \bar{c}) < w(c^1) - w(\bar{c}). \qquad (6.26)$$

Also, from (iii), substituting $[(\bar{c}/2) + (c^1/2)]$ for c, we obtain

$$w(\bar{c}) - \bar{p}\bar{c} \geqslant w((\bar{c}/2) + (c^1/2)) - \bar{p}((\bar{c}/2) + (c^1/2)).$$

Hence,

$$\bar{p}(c^1 - \bar{c}) \geqslant 2[w((\bar{c}/2) + (c^1/2)) - w(\bar{c})]. \qquad (6.27)$$

From (6.26) and (6.27) we obtain

$$\begin{aligned} (\bar{p} - p^s)(c^1 - \bar{c}) &> 2[w((\bar{c}/2) + (c^1/2)) - w(\bar{c})] - [w(c^1) - w(\bar{c})] \\ &\geqslant \varepsilon_1 \qquad \text{[from (6.24), since } c^1 \text{ is in } B]. \end{aligned}$$

Therefore, $0<\varepsilon_1 \leqslant \| \bar{p}-p^s \| \, \| c^1-\bar{c} \| = \| \bar{p}-p^s \| \varepsilon_0$. Hence, for $s \geqslant 0$, $\| \bar{p}-p^s \| \geqslant \varepsilon_1/\varepsilon_0 > 0$. This contradicts (i) and, therefore, completes the proof of the lemma. ∎

Lemma 6 now helps us to establish the turnpike property of optimal programs (Proposition 6) under the maintained assumptions (A.1)–(A.5).

Proof of Proposition 6

Since $\langle \bar{x}(t), \bar{y}(t) \rangle$ is an optimal program from $\tilde{y} \gg 0$, there is (by Proposition 3) a sequence $\langle \bar{p}(t) \rangle$ such that $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is a competitive program.

By Lemma 2, $[\bar{x}(t), \bar{y}(t), w(\bar{c}(t)), \bar{p}(t)] \gg 0$ for $t \geqslant 0$, and also for $t \geqslant 0$

$$0 = \bar{p}(t+1) - \bar{p}(t) A \qquad \text{for} \quad t \geqslant 0. \tag{6.28}$$

From (6.28) it follows that

$$\hat{\lambda}^t \bar{p}(t) = \bar{p}(0) \hat{\lambda}^t A^t \qquad \text{for} \quad t \geqslant 0. \tag{6.29}$$

Now, $\hat{\lambda}^t A^t$ converges to a matrix $\bar{A}$, such that $\bar{a}_{ij} = \hat{p}_j \hat{y}_i$ (Karlin [7, p. 249]) for $i = 1, ..., n; j = 1, ..., n$. Thus, we have

$$\hat{\lambda}^t \bar{p}(t) \to [\bar{p}(0) \hat{y}] \hat{p} \qquad \text{as} \quad t \to \infty. \tag{6.30}$$

Define $\mu = 1/[\bar{p}(0) \hat{y}]^{1/\alpha}$, and a sequence $\langle x'(t), y'(t) \rangle$ by $y'(t) = \mu y^*(t)$, $x'(t) = \mu x^*(t)$ for $t \geqslant 0$, where $\langle x^*(t), y^*(t) \rangle$ is the steady-state optimal program of Proposition 5. Also, define $\langle p'(t) \rangle$ by $p'(t) = \mu^\alpha p^*(t)$ for $t \geqslant 0$ and $\langle c'(t) \rangle$ by $c'(t) = \mu c^*(t)$ for $t \geqslant 0$, where $c^*(t)$ is the consumption sequence associated with $\langle x^*(t), y^*(t) \rangle$ and $\langle p^*(t) \rangle$ is the corresponding sequence of present value prices. Now $\langle x^*(t), y^*(t), p^*(t) \rangle$ is competitive, and $\mu > 0$ since $\bar{p}(0) \gg 0$ by Lemma 2. Therefore, by Lemma 3(b), $\langle x'(t), y'(t), p'(t) \rangle$ is competitive. Moreover, $\lim_{t \to \infty} p'(t) x'(t) = \lim_{t \to \infty} \mu^{1-\alpha} p^*(t) x^*(t) = 0$. Hence $\langle x'(t), y'(t) \rangle$ is optimal. Clearly it is a steady-state program with consumption sequence $\langle c'(t) \rangle$. Therefore $\langle x'(t), y'(t) \rangle$ is a steady-state optimal program. This establishes (i).

Now, $p'(0) = \mu^{-\alpha} p^*(0) = [\bar{p}(0) \hat{y}] \hat{p}$. Therefore, from (6.30), we have

$$\hat{\lambda}^t \bar{p}(t) \to p'(0) \qquad \text{as} \quad t \to \infty. \tag{6.31}$$

Since $\langle x'(t), y'(t), p'(t) \rangle$ is competitive, we have

$$w(c'(0)) - p'(0) c'(0) \geqslant w(c) - p'(0) c \qquad \text{for all } c \text{ in } R^n_+. \tag{6.32}$$

Since $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is competitive, Lemma 4 yields

$$w(\bar{c}(t)/g^t) - \hat{\lambda}^t \bar{p}(t)(\bar{c}(t)/g^t) \geqslant w(c) - \hat{\lambda}^t \bar{p}(t) c \qquad \text{for all } c \text{ in } R^n_+, \ t \geqslant 0. \tag{6.33}$$

By virtue of (6.31), (6.32), and (6.33) we may appeal to Lemma 6 and conclude that

$$\lim_{t \to \infty} (\bar{c}(t)/g^t) = c'(0). \tag{6.34}$$

Now, in view of the "principle of optimality," for $s \geq 0$, we have $\langle \bar{x}(t+s), \bar{y}(t+s) \rangle$ is optimal from $\bar{y}(s)$. So, for $s \geq 0$, we have $\langle \bar{x}(t+s), \bar{y}(t+s) \rangle$ is efficient from $\bar{y}(s)$. Hence, using Result 1, we have, for $s \geq 0$,

$$\bar{y}(s) = \sum_{t=0}^{\infty} A^t \bar{c}(s+t) = \sum_{t=0}^{\infty} g^t A^t [\bar{c}(s+t)/g^t].$$

So, we obtain for $s \geq 0$,

$$[\bar{y}(s)/g^s] = \sum_{t=0}^{\infty} g^t A^t [\bar{c}(s+t)/g^{t+s}].$$

For the steady-state optimal program $\langle x'(t), y'(t) \rangle$, we have

$$y'(0) = \mu y^* = \mu(I - gA)^{-1} c^* = (I - gA)^{-1} c'(0) = \sum_{t=0}^{\infty} g^t A^t c'(0).$$

So, we obtain, for $s \geq 0$,

$$[\{\bar{y}(s)/g^s\} - y'(0)] = \sum_{t=0}^{\infty} g^t A^t [\{\bar{c}(s+t)/g^{t+s}\} - c'(0)]. \tag{6.35}$$

Now, given any $\varepsilon > 0$, there is, by (6.34), a positive integer N^*, such that $N \geq N^*$ implies $\| \{\bar{c}(N)/g^N\} - c'(0) \| \leq \varepsilon$. Using this in (6.35), we have for $s \geq N^*$

$$[\{\bar{y}(s)/g^s\} - y'(0)] \leq \varepsilon \sum_{t=0}^{\infty} g^t A^t e = \varepsilon (I - gA)^{-1} e \qquad [\text{since } gA \geq 0]. \tag{6.36}$$

Similarly, by (6.35), for $s \geq N^*$,

$$[\{\bar{y}(s)/g^s\} - y'(0)] \geq -\varepsilon \sum_{t=0}^{\infty} g^t A^t e = -\varepsilon (I - gA)^{-1} e. \tag{6.37}$$

Thus, for $s \geq N^*$,

$$\| \{\bar{y}(s)/g^s\} - y'(0) \| \leq \varepsilon \, \| (I - gA)^{-1} e \| \tag{6.38}$$

which proves that $\{\bar{y}(s)/g^s\} \to y'(0)$ as $s \to \infty$. This establishes (ii). ∎

Proof of Theorem 1

We will show that (5.1) implies

$$\liminf_{t \to \infty} \bar{p}(t)\, \bar{y}(t) = 0 \tag{6.39}$$

so that the optimality of $\langle \bar{x}(t), \bar{y}(t) \rangle$ then follows by Proposition 1.

To this end, our first objective is to show that

$$\bar{p}(0)\, \hat{y} > \hat{p}\hat{y}. \tag{6.40}$$

We start by noting that if y is in Y, then $w(c^*) - w(y/\theta) \geqslant \hat{p}c^* - \hat{p}(y/\theta)$ [using (6.17)] $= 0$ [using the definition of Y]. So $w(c^*) \geqslant w(y/\theta) = w(y)/\theta^{1-\alpha}$ and so $\theta^{1-\alpha}w(c^*) \geqslant w(y)$. Define $a = \theta^{1-\alpha}$, and $[(1/a^{\alpha}) - 1] = b$. Note that since $0 < \theta < 1$, so $0 < a < 1$, and $b > 0$. Since $\bar{y}(0)$ is in Y, therefore, $aw(c^*) \geqslant w(\bar{y}(0))$.

Now, since $\langle \bar{x}(t), \bar{y}(t) \rangle$ is competitive $w(\bar{c}(0) + a \| c^* \| e) - w(\bar{c}(0)) \leqslant \bar{p}(0)\, a \| c^* \| e = a \| c^* \| \, \| \bar{p}(0) \|$. Also, using the fact that $\bar{c}(0) \leqslant \bar{y}(0)$, and so $w(\bar{c}(0)) \leqslant w(\bar{y}(0))$, we have $w(\bar{c}(0) + a \| c^* \| e) - w(\bar{c}(0)) \geqslant w(a \| c^* \| e) - w(\bar{y}(0)) \geqslant w(ac^*) - aw(c^*) = a^{1-\alpha}w(c^*) - aw(c^*) = a[(1/a^{\alpha}) - 1]\, w(c^*) = ab\, w(c^*)$. Using the above two results, we conclude that

$$\| \bar{p}(0) \| \geqslant bw(c^*)/\| c^* \|. \tag{6.41}$$

This yields $\bar{p}(0)\, \hat{y} \geqslant m(\hat{y})\, \bar{p}(0)\, e = m(\hat{y}) \| \bar{p}(0) \| \geqslant m(\hat{y})\, bw(c^*)/\| c^* \| > m(\hat{y})\, b\hat{p}c^*/\| c^* \|$ [using (6.15) and (6.17)] $\geqslant m(\hat{y})\, b\hat{p}\hat{y}/E \| c^* \|$ (using the definition of E). Summarizing these inequalities, we have

$$\bar{p}(0)\, \hat{y} > (\hat{p}\hat{y})\, [bm(\hat{y})/E \| c^* \|] = \hat{p}\hat{y}b\eta. \tag{6.42}$$

Using the definition of θ, $\theta^{\alpha(1-\alpha)} = \eta/[\eta + 1]$, so that we have $[1/\theta^{\alpha(1-\alpha)}] = 1 + (1/\eta)$. This means that $\{[1/\theta^{\alpha(1-\alpha)}] - 1\}\ \eta = 1$; or, using the notation $a = \theta^{1-\alpha}$, and $[(1/a^{\alpha}) - 1] = b$, we have $b\eta = 1$. Using this last piece of information in (6.42) yields (6.40).

Now, suppose, contrary to (6.39), that $\liminf_{t \to \infty} \bar{p}(t)\, \bar{y}(t) > 0$. Then, there is $\mu > 0$, and an integer $T_1 \geqslant 0$, such that $t \geqslant T_1$ implies

$$\bar{p}(t)\, \bar{y}(t) \geqslant \mu. \tag{6.43}$$

Since $\langle \bar{x}(t), \bar{y}(t), \bar{p}(t) \rangle$ is competitive, therefore, by Lemma 2, we have

$$\bar{p}(t+1) = \bar{p}(t)\, A \qquad \text{for} \quad t \geqslant 0. \tag{6.44}$$

Using (6.44), we note that for $t \geqslant 0$,

$$\hat{\lambda}^{t+1}\bar{p}(t+1)\, \hat{y} = \hat{\lambda}^{t+1}\bar{p}(t)\, A\hat{y} = \hat{\lambda}^{t}\bar{p}(t)\, \hat{y}$$

and so we have

$$\bar{\lambda}^t \bar{p}(t)\,\hat{y} = \bar{p}(0)\,\hat{y}. \tag{6.45}$$

Similarly, using (6.20), we have

$$\hat{\lambda}^t p^*(t)\,\hat{y} = \hat{p}\hat{y}. \tag{6.46}$$

Denoting $[\bar{p}(0)\,\hat{y} - \hat{p}\hat{y}]$ by ε, and noting by (6.40) that $\varepsilon > 0$, we have, by (6.45) and (6.46),

$$\hat{\lambda}^t[\bar{p}(t) - p^*(t)]\,\hat{y} = \varepsilon. \tag{6.47}$$

Now, in view of (6.44), we know that

$$\hat{\lambda}^t \bar{p}(t) = \bar{p}(0)\,\hat{\lambda}^t A^t \qquad \text{for} \quad t \geqslant 0. \tag{6.48}$$

Since $\hat{\lambda}^t A^t$ converges to a matrix $\bar{A}$, such that $\bar{a}_{ij} = \hat{p}_j \hat{y}_i$ (Karlin [7, p. 249]), we have

$$\hat{\lambda}^t \bar{p}(t) \to [\bar{p}(0)\,\hat{y}]\,\hat{p} \qquad \text{as} \quad t \to \infty. \tag{6.49}$$

On the other hand, $\hat{\lambda}^t p^*(t) = \hat{p}$; so, using $\hat{p}\hat{y} = 1$ we can write

$$\hat{\lambda}^t p^*(t) = [\hat{p}\hat{y}]\,\hat{p} \qquad \text{for} \quad t \geqslant 0. \tag{6.50}$$

Combining (6.49) and (6.50), we can find $T_2 \geqslant T_1$, such that

$$\hat{\lambda}^t[\bar{p}(t) - p^*(t)] \geqslant (\varepsilon/2)\,\hat{p} \qquad \text{for} \quad t \geqslant T_2. \tag{6.51}$$

Now $y^*(t) = y^* g^t$ for $t \geqslant 0$, where $0 < g < \hat{\lambda}$. So, in view of (6.49), (6.50), we certainly have $[\bar{p}(t) - p^*(t)]\,y^*(t) \to 0$ as $t \to \infty$. Defining $v = [\mu\varepsilon/8\bar{p}(0)\,\hat{y}]$ we can find $T_3 \geqslant T_2$, such that

$$[\bar{p}(t) - p^*(t)]\,y^*(t) \leqslant v \qquad \text{for} \quad t \geqslant T_3. \tag{6.52}$$

Using this in (5.2) we have for $t \geqslant T_3$

$$[\bar{p}(t) - p^*(t)]\,\bar{y}(t) \leqslant v. \tag{6.53}$$

Combining this with (6.51) yields

$$(\varepsilon/2)\,\hat{p}\bar{y}(t)/\hat{\lambda}^t \leqslant v \qquad \text{for} \quad t \geqslant T_3. \tag{6.54}$$

Relying on (6.49) again, we can find $T_4 \geqslant T_3$, such that $t \geqslant T_4$ implies

$$\hat{\lambda}^t \bar{p}(t) \leqslant 2[\bar{p}(0)\,\hat{y}]\,\hat{p}. \tag{6.55}$$

Using this in (6.54), we have, for $t \geqslant T_4$,

$$\bar{p}(t)\,\bar{y}(t) < 2[\bar{p}(0)\,\hat{y}]\,\hat{p}\bar{y}(t)/\hat{\lambda}' < \{4\bar{p}(0)\,\hat{y}/\varepsilon\}\;v = (\mu/2), \tag{6.56}$$

which contradicts (6.43) and establishes (6.39), and hence the theorem. ∎

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